Weak solutions for the Falk model system of shape memory alloys in energy class (Harmonic Analysis and Nonlinear Partial Differential Equations)

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Weak solutions for the Falk model system of shape memory alloys in energy class

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1 Introduction

This note is based on [19]. We study the initial value problem with periodic boundary conditions of the following Boussinesq-heat system:

$$u_{tt}+u_{xxxx}=(f_1(u_x)\theta+f_2(u_x))_x, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{T}, \tag{1.1}$$
$$\theta_t-\theta_{xx}=f_1(u_x)\theta u_{xt}, \tag{1.2}$$
$$u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), \quad \theta(0,x)=\theta_0(x), \tag{1.3}$$

where $\mathbb{R}^+=\{t\in\mathbb{R}\mid t>0\}$ and $\mathbb{T}=\mathbb{R}/\mathbb{Z}$.

This system describes the dynamics of first order martensitic phase transitions occurring in a ring made of shape memory alloys, where u denotes the longitudinal displacement of the ring, and θ is the temperature. For more details of the Falk model system, we refer the reader to Chapter 5 in the literature [5].

Before stating our results, let us first recall some results related to this paper. Sprekels and Zheng [13] proved the unique global existence of smooth solution for (1.1)-(1.3). In [6], Bubner and Sprekels established unique global existence results of (1.1)-(1.3) for data $(u_0,u_1,\theta_0)\in H^3\times H^1\times H^1$, and discussed the optimal control problem in the case

$$f_1(r)=-r \text{ and } f_2(r)=r^5-r^3+r. \tag{A0}$$

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T. Aiki [1] proved unique global existence of solution with $(u_0, u_1, \theta_0) \in H^3 \times H^1 \times H^1$ for more general nonlinearity, that is,

$$f_1, f_2 \in C^2(\mathbb{R}), \tag{A1}$$

and

$$F_2(r) \geq -C \text{ for } r \in \mathbb{R}, \tag{A2}$$

where $F_2(r)$ is the primitive of $f_2(r)$. We note that the condition (A0) implies the conditions (A1) and (A2).

Systems related to (1.1)-(1.3) have been studied for the case of viscous materials, that is,

$$u_{tt} + u_{xxxx} - u_{xxt} = (f_1(u_x)\theta + f_2(u_x))_x, \tag{1.4}$$

$$\theta_t - \theta_{xx} - |u_{xt}|^2 = f_1(u_x)\theta u_{xt}. \tag{1.5}$$

The viscosity term simplifies the analysis because this term has smoothing property. In fact, K.-H. Hoffman and Zochowski establish the existence result decomposing (1.4) into a system of two parabolic equations in [8]. Sprekels, Zheng and Zhu [14] prove the asymptotic behavior of the solution for (1.4)-(1.5) as $t \to \infty$. However, the literature [5] says that there is no interior friction from the experimental evidence. Moreover, it seems that for (1.1)-(1.4) has not been determined the asymptotic behavior of the solution as $t \to \infty$. Another interesting property of shape memory alloys is *hysteresis*. There are a lot of models and results from this point of view. For related results to hysteresis, we refer to e.g. [2].

System (4.1)-(4.2) conserves the energy, namely, the integral

$$E(u(t), u_t(t), \theta(t)) = \frac{1}{2}(\|u_t\|_{L^2_x}^2 + \|u_{xx}\|_{L^2_x}^2) + \int_0^1 \theta dx + \int_0^1 F_2(u_x)dx \tag{1.6}$$

does not depend on the time t. Therefore, the energy class of this system is $H^2 \times L^2 \times L^1$. In the author's master thesis [18], the unique global existence theorem in $H^2 \times L^2 \times L^2$ is proved, which is slightly smaller than the energy space. When we consider the solvability of (4.1)-(4.4), the energy class seems most natural. Nevertheless, there have been no papers on the solvability of (4.1)-(4.4) in the energy class up to the present. The aim of this paper is to prove the unique global existence of solution for (4.1)-(4.4) in this space. Here the spaces $W^{m,p}$ and H^m are the standard Sobolev spaces, that is, $W^{m,p}$ is equipped with the norm

$$\|f\|_{W^{m,p}} = \sum_{0 \leq k \leq m} \|\partial_x^k f\|_{L^p},$$

and $H^m = W^{m,2}$.

Our main results in this paper are stated as follows:

Theorem 1.1 (Local existence and uniqueness). *Assume that f_1, f_2 satisfy the condition (A1). Let any $\varepsilon \in (0, 1/6)$ be fixed. Then for any $(u_0, u_1, \theta_0) \in H^2 \times L^2 \times L^1$, there exists $T = T(\|u_0\|_{H^2}, \|u_1\|_{L^2}, \|\theta_0\|_{L^1}) > 0$ such that the problem (1.1)-(1.3) has a unique solution (u, θ) on the time interval $[0, T]$, satisfying*

$$\begin{aligned}
u &\in C([0,T]; H^2(\mathbb{T})) \cap L^4(0,T; W^{2,4}(\mathbb{T})),\\
u_t &\in L^\infty(0,T; L^2(\mathbb{T})) \cap L^4(0,T; L^4(\mathbb{T})),\\
\theta &\in C([0,T]; L^1(\mathbb{T})),\\
\theta_x &\in L^{\frac{4}{3}+\varepsilon}(0,T; L^{\frac{4}{3}+\varepsilon}(\mathbb{T})).
\end{aligned}$$

Our main tools of the proof of this theorem are the maximal regularity estimate and the Strichartz estimate. In general, the derivative of a solution for parabolic equations is less regular than the right-hand side of the corresponding equations. We call the estimate such a loss of regularity does not occur "maximal regularity". For this estimate, we refer to [3], [9] and [10]. The Strichartz estimate established in [15] is closely related to the restriction theory of the Fourier transform to surfaces and used often in various areas of the study of nonlinear wave equations. For the application of this estimate, we refer to [11], [12], [16] and [17]. Corresponding results in the spatially periodic setting are established by J. Bourgain [4], and more transparent version is given by Fang and Grillakis in [7].

Combining this result with the energy conservation law, we obtain the following global result.

Theorem 1.2 (Global existence). *In addition to the assumptions of Theorems 1.1, suppose that (A2) and $\theta_0 \geq 0$. Then, the solution given by Theorems 1.1 can be extended globally in time.*

Remark. We note that the nonlinear term of (1.2) is rewritten as the following form:

$$f_1(u_x)\theta u_{tx} = (f_1(u_x)\theta u_t)_x - f_1'(u_x)u_{xx}\theta u_t - f(u_x)\theta_x u_t,$$

which makes sense in the distribution class.

2 Preliminary Estimates

In this section, we shall summarize several lemmas to be used in the proof of Theorem 1.1. The key estimates for this result are a space-time estimate for the free solution of (1.1) (the so-called Strichartz estimate) and the maximal regularity estimate of (1.2).

Before the stating these estimates, we introduce several notations. We write a partial derivative with respect to a variable y as follows:

$$\partial_y := \frac{\partial}{\partial y},$$

and the Lebesgue norm is defined as

$$\|f; L_T^p L_x^q\| = \|f\|_{L_T^p L_x^q} := \left\{ \int_0^T \left(\int_{\mathbb{T}} |f(x,t)|^q dx \right)^{\frac{p}{q}} dt \right\}^{\frac{1}{p}},$$

in particular

$$\|f; L_{T,x}^p\| = \|f\|_{L_{T,x}^p} := \|f\|_{L_T^p L_x^p}.$$

For an 1-parameter (semi-)group $V(t)$, we write

$$\Gamma(V)f := \int_0^t V(t-s)f(s)ds.$$

Throughout this paper, C ($C(r)$) is a positive constant (depending only on r) and q' and q satisfy the relation $1/q + 1/q' = 1$.

Proposition 2.1 (Strichartz type estimate [4], [7]). *The following estimates holds,*

$$\|V_\pm(\cdot)g; L_{T,x}^4\| \le C\|g; L_x^2\|, \tag{2.1}$$

$$\left\|\Gamma(V_\pm)f; L_{T,x}^4\right\| \le C\|f; L_{T,x}^{\frac{4}{3}}\|, \tag{2.2}$$

and

$$\left\|\Gamma(V_\pm)f; L_T^\infty L_x^2\right\| \le C\left\|f; L_{T,x}^{\frac{4}{3}}\right\|, \tag{2.3}$$

where $V_\pm := e^{\pm it\partial_x^2}$.

Proposition 2.2 (Maximal regularity). *For any* $q \in (1,\infty)$, *we have*

$$\|\partial_x^2\Gamma(U)f; L_{T,x}^q\| \le C(1+T)\|f; L_{T,x}^q\|, \tag{2.4}$$

where $U(t) := e^{t\partial_x^2}$.

Proof. For the case the space is $\mathbb{R}$, we can find this estimate in [9] and [10]. By using cutoff function argument, we can prove 2.4. □

The following estimates are the standard estimates of the heat equation.

Proposition 2.3 (L^p-L^q Estimate). *For any $1 \le q \le p \le \infty$ and $t > 0$, we have*

$$\|U(t)g; L^p_x\| \le C\left(1 + \frac{1}{t^{1/2(1/q-1/p)}}\right)\|g; L^q_x\|, \tag{2.5}$$

and

$$\|\partial_x U(t)g; L^p_x\| \le \frac{C}{t^{1/2}}\left(1 + \frac{1}{t^{1/2(1/q-1/p)}}\right)\|g; L^q_x\|. \tag{2.6}$$

Remark. In this paper, since global time estimates are not needed, we may regard these estimates as the following well-known form

$$\|U(t)g; L^p_x\| \le \frac{C}{t^{1/2(1/q-1/p)}}\|g; L^q_x\|,$$

and

$$\|\partial_x U(t)g; L^p_x\| \le \frac{C}{t^{1/2+1/2(1/q-1/p)}}\|g; L^q_x\|.$$

In the end of this section we formulate estimates obtained by using the Gagliardo-Nirenberg inequality. We make frequent use of the following lemma in this paper.

Lemma 2.1. *For any $p < \frac{5+6\varepsilon}{3}q'$ there exist $C(p,q,\varepsilon) > 0$ such that*

$$\|\theta; L^p_T L^q_x\| \le CT^{C(p,q,\varepsilon)}\|\theta; L^\infty_T L^1_x\|^{1-\sigma(\varepsilon)}\|\theta_x; L^{\frac{4}{3}+\varepsilon}_{T,x}\|^{\sigma(\varepsilon)}, \tag{2.7}$$

where $\sigma(\varepsilon) = \frac{1}{q'}\left(\frac{4+3\varepsilon}{5+6\varepsilon}\right)$ and $C(p,q,\varepsilon) = \frac{1}{p} - \frac{1}{q'}\frac{3}{5+6\varepsilon}$.

In particular, for any $p < \frac{2(5+6\varepsilon)}{3}$ there exist $C(p,\varepsilon) > 0$ such that

$$\|\theta; L^p_T L^2_x\| \le CT^{C(p,\varepsilon)}\|\theta; L^\infty_T L^1_x\|^{1-\sigma_2(\varepsilon)}\|\theta_x; L^{\frac{4}{3}+\varepsilon}_{T,x}\|^{\sigma_2(\varepsilon)}, \tag{2.8}$$

where $\sigma_2(\varepsilon) = \frac{4+3\varepsilon}{2(5+6\varepsilon)}$ and $C(p,\varepsilon) = \frac{1}{p} - \frac{3}{2(5+6\varepsilon)}$.

Proof. We only prove (2.8). By the Gagliardo-Nirenberg inequality we have

$$\|\theta; L^2_x\| \le C\|\theta; L^1_x\|^{\frac{6+9\varepsilon}{2(5+6\varepsilon)}}\|\theta_x; L^{\frac{4}{3}+\varepsilon}_x\|^{\frac{4+3\varepsilon}{2(5+6\varepsilon)}}.$$

Therefore, we have

$$\begin{aligned}
\|\theta; L^p_T L^2_x\| &\le C\left\|\|\theta; L^1_x\|^{\frac{6+9\varepsilon}{2(5+6\varepsilon)}}\|\theta_x; L^{\frac{4}{3}+\varepsilon}_x\|^{\frac{4+3\varepsilon}{2(5+6\varepsilon)}}; L^{\frac{4}{3}+\varepsilon}_T\right\| \\
&\le C\|\theta\|^{\frac{6+9\varepsilon}{2(5+6\varepsilon)}}_{L^\infty_T L^1_x}\left\|\|\theta_x; L^{\frac{4}{3}+\varepsilon}_x\|^{\frac{4+3\varepsilon}{2(5+6\varepsilon)}}; L^{\frac{4}{3}+\varepsilon}_T\right\| \\
&\le C\|\theta\|^{\frac{6+9\varepsilon}{2(5+6\varepsilon)}}_{L^\infty_T L^1_x}\|\theta_x; L^{\frac{p(4+3\varepsilon)}{2(5+6\varepsilon)}}_T L^{\frac{4}{3}+\varepsilon}_x\|^{\frac{4+3\varepsilon}{2(5+6\varepsilon)}}.
\end{aligned}$$

By the assumption:

$$\frac{p(4+3\varepsilon)}{2(5+6\varepsilon)} < \frac{4}{3} + \varepsilon,$$

there holds

$$\|\theta_x; L_T^{p\frac{4+3\varepsilon}{2(5+6\varepsilon)}} L_x^{\frac{4}{3}+\varepsilon}\| \leq CT^{\frac{2(5+6\varepsilon)-3p}{p(4+3\varepsilon)}} \|\theta_x; L_{T,x}^{\frac{4}{3}+\varepsilon}\|,$$

which completes the proof. □

3 Local Existence Theorem

Proof of Theorem 1.1. In this section we prove the local existence for the problem (1.1)-(1.3). We will denote by $\widehat{f}$ the Fourier coefficient of the function f with respect to the space variable, i.e.

$$\widehat{f} = \int_{\mathbb{T}} e^{-2\pi i x k} f(x) dx.$$

Without loss of generality we may assume $T < 1$. We first restate the equation (1.1). For simplicity, we write $F := (f_1(v)\theta + f_2(v))$. Since $\widehat{F}(0)$ does not depend on x, (3.3) can be rewritten as follows:

$$u_{tt} + u_{xxxx} = \{F - \widehat{F}(0)\}_x. \tag{3.1}$$

Let

$$v := u_x. \tag{3.2}$$

Differentiating both sides of (3.1), sufficiently smooth solutions satisfy

$$v_{tt} + v_{xxxx} = \{F - \hat{F}(0)\}_{xx}. \tag{3.3}$$

Here for any f such that $\widehat{f}(0) = 0$, we define ∂_x^{-2} as follows

$$\partial_x^{-2} f(x) := -\sum_{k \neq 0} \frac{e^{2\pi i k x}}{(2\pi k)^2} \widehat{f}(k).$$

We note that by (3.2), $\widehat{v}(0) = 0$ and $\widehat{v}_t(0) = 0$.
Putting

$$v^{\pm} := v \pm i\partial_x^{-2} v_t,$$

we have

$$\begin{aligned}\partial_t v &= v_t \pm i\partial_x^{-2} v_{tt} \\ &= v_t \pm i\partial_x^{-2}\left\{-(\partial_x)^4 v + \partial_x^2(F - \widehat{F}(0))\right\} \\ &= v_t \mp iAv \pm i(F - \widehat{F}(0)) \\ &= \mp i\partial_x^2(v \pm i\partial_x^{-2}v_t) \pm i(F - \widehat{F}(0)).\end{aligned} \tag{3.4}$$

Then (3.3) is made into the following two Schrödinger type equations

$$\partial_t v^{\pm} = \mp i\partial_x^2 v^{\pm} \pm i\{F - \widehat{F}(0)\}.$$

Notice that since

$$v_t = \frac{\partial_x^2}{2i}(v^+ - v^-), \tag{3.5}$$

this transformation is useful for the estimate of v_t.

We first show the time local existence and uniqueness of solution (v^+, v^-, θ) with $\widehat{v}^{\pm}(0) = 0$ in the space $H^1 \times H^1 \times L^1$. For $A > 0$, we define the space

$$X_A^T = \{(v^+, v^-, \theta) |\ \|(v^+, v^-, \theta)\|_X := \|v^+\|_a + \|v^-\|_a + \|\theta\|_b \le A\},$$

where

$$\begin{aligned}\|v^{\pm}\|_a &:= \|v^{\pm}; L_T^\infty H_x^1\| + \|\partial_x v^{\pm}; L_{T,x}^4\|, \\ \|\theta\|_b &:= \|\theta; L_T^\infty L_x^1\| + \|\partial_x \theta; L_{T,x}^{4/3+\varepsilon}\|.\end{aligned}$$

and the operator $\Lambda : (v^+, v^-, \theta) \mapsto (\Lambda_+ v^+, \Lambda_- v^-, \Lambda_b \theta)$ as follows:

$$\Lambda_{\pm} v^{\pm} = V_{\pm}(t)v^{\pm}(0) \pm i\Gamma(V_{\pm})(F - \widehat{F}(0)), \tag{3.6}$$

$$\Lambda_b \theta = U(t)\theta_0 + \Gamma(U)(f_1(u_x)\theta u_{tx}). \tag{3.7}$$

We shall prove that for an appropriate choice of A and T, the operator Λ is a contraction of X_A^T into itself. We note that $v^{\pm}(0, x)$ and $(F - \widehat{F}(0))$ have average zero, therefore, so do $\Lambda_{\pm} v^{\pm}$.

We show estimates for (3.6).

For the linear part, by (2.1) we have

$$\begin{aligned}\|V_{\pm}(\cdot)v^{\pm}(0)\|_a &\le \|V_{\pm}(\cdot)v^{\pm}(0); L_T^\infty H_x^2\| + \|\partial_x^2 V_{\pm}(\cdot)v^{\pm}(0); L_{T,x}^4\| \\ &\le C(\|u_0\|_{H^2} + \|u_1\|_{L^2}).\end{aligned}$$

Since by the embedding inequality $\|v; L_x^\infty\| \le C$, we have for $i = 1, 2$ and $j = 0, 1, 2$

$$\|f_i^{(j)}(v); L_{T,x}^\infty\| \le C. \tag{3.8}$$

For the nonlinear part, by (2.2), (2.3), (2.8) and (3.8) we have

$$
\begin{aligned}
\|\Gamma(V_\pm)(F-\widehat{F}(0))\|_a &\le \|\Gamma(V_\pm)(F-\widehat{F}(0)); L_T^\infty H_x^1\| + \|\partial_x\Gamma(V_\pm)(F-\widehat{F}(0)); L_{T,x}^4\| \\
&\le C\|\partial_x(f_1(v)\theta + f_2(v)); L_{T,x}^{\frac{4}{3}}\| \\
&\le C\|f_1'(v)v_x\theta + f_1(v)\theta_x + f_2'(v)v_x; L_{T,x}^{\frac{4}{3}}\| \\
&\le C\|v_x; L_{T,x}^4\|\|\theta; L_{T,x}^2\| + CT^{C(\varepsilon)}\|\theta_x; L_{T,x}^{\frac{4}{3}+\varepsilon}\| + CT^{1/2}\|v_x; L_{T,x}^4\| \\
&\le CT^{C(\varepsilon)}\|v_x; L_{T,x}^4\|\|\theta; L_T^\infty L_x^1\|^{1-\sigma_2(\varepsilon)}\|\theta_x; L_{T,x}^{\frac{4}{3}+\varepsilon}\|^{\sigma_2(\varepsilon)} \\
&\quad + CT^{C(\varepsilon)}\|\theta_x; L_{T,x}^{\frac{4}{3}+\varepsilon}\| + CT^{1/2}\|v_x; L_{T,x}^4\|.
\end{aligned}
$$

Therefore we obtain the following estimate,

$$
\|\Lambda_a v^\pm\| \le C(\|u_0\|_{H^2} + \|u_1\|_{L^2}) + CT^{C(\varepsilon)}(\|v^\pm\|_a\|\theta\|_b + \|\theta\|_b + \|v^\pm\|_a). \tag{3.9}
$$

Next we show the estimate for the heat equation (3.7). For the linear part of (3.7), since $\varepsilon < 1/6$ we have

$$
\begin{aligned}
\|U(\cdot)\theta_0\|_b &= \|U(\cdot)\theta_0; L_T^\infty L_x^1\| + \|\partial_x U(\cdot)\theta_0; L_{T,x}^{\frac{4}{3}+\varepsilon}\| \\
&\le C\|\theta_0\|_{L_x^1} + C\|\theta_0\|_{L_x^1}\left(\int_0^T s^{-\left(\frac{5}{6}+\varepsilon\right)}ds\right)^{\frac{1}{4/3+\varepsilon}} \\
&\le C(1 + T^{\left(\frac{1}{6}-\varepsilon\right)\left(\frac{3}{4+3\varepsilon}\right)})\|\theta_0\|_{L_x^1} \le C\|\theta_0\|_{L_x^1}.
\end{aligned}
$$

For the nonlinear term of (3.7), since

$$
u_{tx}\theta f_1(u_x) = (u_t\theta f_1(u_x))_x - u_t\theta_x f_1(u_x) - u_t\theta u_{xx} f_1'(u_x),
$$

we can split the nonlinear term into the six parts as follows:

$$
\begin{aligned}
\|\Gamma(U)(u_{tx}\theta u_x)\|_b &\le \|\Gamma(U)(u_t\theta u_{xx}f_1'(u_x)); L_T^\infty L_x^1\| + \left\|\Gamma(U)(u_t\theta u_{xx}f_1'(u_x))_x; L_{T,x}^{\frac{4}{3}+\varepsilon}\right\| \\
&\quad + \|\Gamma(U)(u_t\theta_x f_1(u_x)) L_T^\infty L_x^1\| + \left\|\Gamma(U)(u_t\theta_x f_1(u_x))_x; L_{T,x}^{\frac{4}{3}+\varepsilon}\right\| \\
&\quad + \|\Gamma(U)(u_t\theta f_1(u_x))_x; L_T^\infty L_x^1\| + \left\|\Gamma(U)(u_t\theta f_1(u_x))_{xx}; L_{T,x}^{\frac{4}{3}+\varepsilon}\right\| \\
&:= I_{1,1} + I_{1,2} + I_{2,1} + I_{2,2} + I_{3,1} + I_{3,2}.
\end{aligned}
$$

Using the Hölder inequality and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
I_{1,2} &\le \left\| \int_0^{\cdot} \frac{C}{(\cdot - s)^{\frac{1}{2}+\frac{1}{2}\left(1-\frac{1}{4/3+\varepsilon}\right)}} \|u_t\theta u_{xx} f_1(u_x)\|_{L^1_x} ds; L_T^{\frac{4}{3}+\varepsilon} \right\| \\
&\le C T^{\frac{2}{3}+\frac{1}{p}} \left\| \int_0^{\cdot} \frac{1}{(\cdot - s)^{1+\frac{1}{4/3+\varepsilon}}} \|u_t\theta u_{xx} f_1(u_x)\|_{L^1_x} ds; L_T^p \right\| \\
&\le C T^{\frac{1}{r}} \|u_t\theta u_{xx} f_1(u_x); L_T^q L_x^1\| \\
&\le C T^{\frac{1}{r}} \|u_t; L_{T,x}^4\| \|u_{xx}; L_{T,x}^4\| \|\theta; L_T^{\frac{2-q}{2q}} L_x^2\|,
\end{aligned}$$

where r, q are any numbers such that $q > 1$ and $\frac{1}{r} + \frac{1}{q} = \frac{9}{2(4+3\varepsilon)}$. Here fixing $q \le \frac{5+6\varepsilon}{4+3\varepsilon} = \frac{5}{4} + \frac{9\varepsilon}{3(3+3\varepsilon)}$, by (2.8) we obtain

$$I_{1,2} \le C T^{\frac{1}{r}+C(q,\varepsilon)} \|u_t; L_{T,x}^4\| \|u_{xx}; L_{T,x}^4\| \|\theta; L_T^\infty L_x^1\|^{1-\sigma_2(\varepsilon)} \|\theta_x; L_{T,x}^{\frac{4}{3}+\varepsilon}\|^{\sigma_2(\varepsilon)}.$$

In the same way, for $I_{2,2}$ we have

$$I_{2,2} \le C T^{\frac{1}{\tilde r}} \|u_t\theta_x; L_T^{\tilde q} L_x^1\|,$$

where $\tilde r$ and $\tilde q > 1$ satisfy $\frac{1}{\tilde r} + \frac{1}{\tilde q} = \frac{9}{2(4+3\varepsilon)}$. Putting $\tilde q = 1 + \frac{9\varepsilon}{16+3\varepsilon}$, we have $\tilde r = \frac{4(4+3\varepsilon)}{2-3\varepsilon}$ and

$$\begin{aligned}
\|u_t\theta_x; L_T^{1+\frac{9\varepsilon}{16+3\varepsilon}} L_x^1\| &\le \|u_t; L_{T,x}^4\| \|\theta_x; L_T^{\frac{4}{3}+\varepsilon} L_x^{\frac{4}{3}}\| \\
&\le \|u_t; L_{T,x}^4\| \|\theta_x; L_{T,x}^{\frac{4}{3}+\varepsilon}\|.
\end{aligned}$$

Therefore we have

$$I_{2,2} \le C T^{\frac{2-3\varepsilon}{4(4+3\varepsilon)}} \|u_t; L_{T,x}^4\| \|\theta_x; L_{T,x}^{\frac{4}{3}+\varepsilon}\|.$$

By these estimates, we immediately have

$$\begin{aligned}
I_{1,1} &\le \|u_t\theta u_{xx} f_1(u_x); L_{T,x}^1\| \\
&\le C \|u_t; L_{T,x}^4\| \|u_{xx}; L_{T,x}^4\| \|\theta; L_{T,x}^2\| \\
&\le C T^{C(2,\varepsilon)} \|u_t; L_{T,x}^4\| \|u_{xx}; L_{T,x}^4\| \|\theta; L_T^\infty L_x^1\|^{1-\sigma_2(\varepsilon)} \|\theta_x; L_{T,x}^{\frac{4}{3}+\varepsilon}\|^{\sigma_2(\varepsilon)}.
\end{aligned}$$

and

$$\begin{aligned}
I_{2,1} &\le \|u_t\theta_x f_1(u_x); L_{T,x}^1\| \\
&\le C T^{\frac{4(4+3\varepsilon)}{9\varepsilon}} \|u_t; L_{T,x}^4\| \|\theta_x; L_T^{\frac{4}{3}+\varepsilon} L_x^{\frac{4}{3}}\| \\
&\le C T^{\frac{4(4+3\varepsilon)}{9\varepsilon}} \|u_t; L_{T,x}^4\| \|\theta_x; L_{T,x}^{\frac{4}{3}+\varepsilon}\|.
\end{aligned}$$

For $I_{3,1}$, by the Hölder inequality and (2.8) we obtain

$$\begin{aligned}
I_{3,1} &\le \left\| \int_0^{\cdot} \frac{C}{(\cdot - s)^{1/2}} \|u_t\theta f_1(u_x)\|_{L^1_x} ds \right\|_{L^\infty_T} \\
&\le C \sup_{t\in[0,T]} \left| \left(\int_0^t (t-s)^{-\frac{p'}{2}} ds \right)^{\frac{1}{p'}} \left(\int_0^t \|u_t\theta f_1(u_x); L^1_x\|^p ds \right)^{\frac{1}{p}} \right| \\
&\le CT^{\frac{1}{2}-\frac{1}{p}} \|u_t\theta f_1(u_x); L^p_T L^1_x\| \\
&\le CT^{\frac{1}{2}-\frac{1}{p}} \|u_t; L^\infty_T L^2_x\| \|\theta; L^p_T L^2_x\| \\
&\le CT^{\frac{1}{2}-\frac{1}{p}+C(p,\varepsilon)} \|u_t; L^\infty_T L^2_x\| \|\theta; L^\infty_T L^1_x\|^{1-\sigma(\varepsilon)} \|\theta_x; L^{\frac{4}{3}+\varepsilon}_{T,x}\|^{\sigma(\varepsilon)}.
\end{aligned}$$

where $p \in (2, \frac{2(5+6\varepsilon)}{3}]$.

Finally, we calculate for the $I_{3,2}$. By (2.4) we have,

$$\begin{aligned}
I_{3,2} &\le C(1+T)\|u_t\theta f_1(u_x); L^{\frac{4}{3}+\varepsilon}_{T,x}\| \\
&\le C(1+T)\|u_t; L^4_{T,x}\| \|\theta; L^{2+\frac{18\varepsilon}{8-3\varepsilon}}_{T,x}\|.
\end{aligned}$$

For $L^{2+\frac{18\varepsilon}{8-3\varepsilon}}_{T,x}$-norm of θ, using (2.7) we have

$$\|\theta; L^{2+\frac{18\varepsilon}{8-3\varepsilon}}_{T,x}\| \le CT^{\frac{3(8-3\varepsilon)}{4(8+15\varepsilon)}} \|\theta; L^\infty_T L^1_x\|^{1-\rho(\varepsilon)} \|\theta_x; L^{\frac{4}{3}+\varepsilon}_{T,x}\|^{\rho(\varepsilon)},$$

where $\rho(\varepsilon) = \frac{8+15\varepsilon}{4(5+6\varepsilon)}$.

Then combining these estimates we obtain the following estimate,

$$\|\Lambda_b\theta\|_b \le C\|\theta_0\|_{L^1_x} + CT^{C(\varepsilon)}(\|\theta\|_b\|v^\pm\|_a^2 + \|\theta\|_b\|v^\pm\|_a). \tag{3.10}$$

Therefore, combining (3.9) and (3.10),

$$\begin{aligned}
\|\Lambda(v^+, v^-, \theta)\|_X &\le C(\|u_0\|_{H^2}, \|u_1\|_{L^2}, \|\theta\|_{L^1}) \\
&\quad + CT^{C(\varepsilon)}(\|v^\pm\|_a + \|v^\pm\|_a\|\theta\|_b + \|v^\pm\|_a^2\|\theta\|_b) \\
&\le C(\|u_0\|_{H^2}, \|u_1\|_{L^2}, \|\theta\|_{L^1}) \\
&\quad + CT^{C(\varepsilon)}(\|(v^+, v^-, \theta)\|_X + \|(v^+, v^-, \theta)\|_X^2 + \|(v^+, v^-, \theta)\|_X^3).
\end{aligned} \tag{3.11}$$

Here by the mean value theorem and (3.8) that for $i = 1, 2$ and $j = 0, 1$

$$\|f_i^{(j)}(v) - f_i^{(j)}(\tilde{v}); L^p\| \le \left\| |v - \tilde{v}| \int_0^1 f_i^{(j+1)}(sv + (1-s)\tilde{v}) ds; L^p \right\| \le C\|v - \tilde{v}; L^p\|.$$

By using this, we obtain

$$\|\Gamma(V^{\pm})\{(f_1(v)\theta - f_1(\tilde{v})\tilde{\theta}) + f_2(v) - f_2(\tilde{v})\}\|_a \leq$$
$$CT^{C(\varepsilon)}\left(\|v-\tilde{v}\|_a\|\theta\|_b + \|v\|_a\|\theta-\tilde{\theta}\|_b + \|v-\tilde{v}\|_a\right),$$

and

$$\|\Gamma(U)(f_1(u_x)\theta u_{tx} - f_1(\tilde{u}_x)\tilde{\theta}\tilde{u}_{tx})\|_b \leq CT^{C(\varepsilon)}\left(\|u\|_a^2\|\theta-\tilde{\theta}\|_b + \|u-\tilde{u}\|_a\|u\|_a\|\theta\|_b\right).$$

Therefore we have

$$\|\Lambda(v^+,v^-,\theta) - \Lambda(\tilde{v}^+,\tilde{v}^-,\tilde{\theta})\|_X \leq CT^{C(\varepsilon)}\left(1 + \|(v^+,v^-,\theta)\|_X + \|(v^+,v^-,\theta)\|_X^2\right)$$
$$\times\ \|(v^+,v^-,\theta) - (\tilde{v}^+,\tilde{v}^-,\tilde{\theta})\|_X. \tag{3.12}$$

Hence it is sufficient to take $A = 2C(\|u_0\|_{H^2}, \|u_1\|_{L^2}, \|\theta\|_{L^1})$ and T such that $CT^{C(\varepsilon)}(1+\|(v^+,v^-,\theta)\|_X + \|(v^+,v^-,\theta)\|_X^2) \leq \frac{1}{2}$ to obtain from (3.11) that Λ maps X_T^A into itself. (3.12) implies that under the same restrictions on A and T, the mapping Λ is a contraction on X_T^A. The contraction mapping principle shows the existence of a unique solution within the ball $\|(v^+,v^-,\theta)\|_X \leq A$. To prove the uniqueness within the whole of the space, it is enough to take T sufficiently small. Then the solution $(v^+,v^-,\theta) \in H^1 \times H^1 \times L^1$ with $\widehat{v}^{\pm}(0) = 0$ is obtained, and this also means the existence of $(v,\theta) \in H^1 \times L^1$ with $\widehat{v}(0) = 0$ because $v = v^+ + v^-$.

Finally we shall verify that the unique existence of $v \in H^1$ leads to that of $u \in H^2$. We can expand v into the trigonometric series as follows:

$$v(x) = \sum_{k\neq 0} \widehat{v}(k)e^{2\pi ikx}.$$

Then if $\widehat{u}(0)$ is obtained, u can be written as

$$u = \sum_{k\neq 0} \frac{\widehat{v}(k)}{2\pi ik}e^{2\pi ikx} + \widehat{u}(0).$$

Obviously the first term of the right hand side converges. The remaining problem is how $\hat{u}(0)$ should be determined. By (1.1),

$$\widehat{u_{tt}}(0) = 0.$$

Therefore we have

$$\widehat{u}(0) = t\widehat{u}_1(0) + \widehat{u}_0(0). \tag{3.13}$$

which yields the desired result.

It is also necessary to show $u \in H^2$. By the Poincaré inequality,

$$\left\|u - \widehat{u}(0); L_x^2\right\| \leq \|v; L_x^2\|.$$

Then by (3.13), we have $\|u; L_x^2\| \leq C$ $(0 \leq t \leq T)$. This means $u \in H^2$.

□

4 Some Remarks

In this section, we give two remarks.

4.1 Initial Boundary Value Problem

We can also consider the following initial boundary problem:

$$u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x, \qquad (t,x) \in \mathbb{R}^+ \times (0,1), \tag{4.1}$$
$$\theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt}, \tag{4.2}$$
$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x), \tag{4.3}$$
$$u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = \theta_x(t,0) = \theta_x(t,1) = 0. \tag{4.4}$$

For this problem we also obtain the following theorem.

Theorem 4.1 (Local existence and uniqueness). *Assume that f_1, f_2 satisfy the condition (A1). Let any $\varepsilon \in (0,1/6)$ be fixed. Then for any $(u_0, u_1, \theta_0) \in H^2 \times L^2 \times L^1$ with $u_0(0) = u_0(1) = 0$, there exists $T = T(\|u_0\|_{H^2}, \|u_1\|_{L^2}, \|\theta_0\|_{L^1}) > 0$ such that the problem* (4.1)-(4.4) *has a unique solution (u,θ) on the time interval $[0,T]$, satisfying*

$$\begin{aligned}
u &\in C([0,T]; H^2(0,1)) \cap L^4(0,T; W^{2,4}(0,1)),\\
u_t &\in L^\infty(0,T; L^2(0,1)) \cap L^4(0,T; L^4(0,1)),\\
\theta &\in C([0,T]; L^1(0,1)),\\
\theta_x &\in L^{\frac{4}{3}+\varepsilon}(0,T; L^{\frac{4}{3}+\varepsilon}(0,1)).
\end{aligned}$$

This theorem can be proved in the same way as Theorem 1.1. Roughly speaking, extending the solutions u and θ of (4.1)-(4.4) as odd and even periodic functions respectively, we regard the initial boundary value problem as the problem with periodic boundary condition.

4.2 Global Existence Theorem

In order to regard the third term of the right hand side of (1.6) as L^1-norm of θ, we need the following proposition related to a sign property for the temperature θ. The proof is obtained by approximating the energy class solution by smooth solutions.

Proposition 4.1 (Maximum principle). *If $\theta_0 \geq 0$ on $\mathbb{T}$ (or $[0,1]$) then the solution θ of* (1.1)-(1.3) *(or*(4.1)-(4.4)*) satisfies $\theta \geq 0$ a.e. on $\mathbb{T}$ (or $[0,1]$) $\times [0,T]$.*

Combining this proposition with the condition (A2) and the energy conservation law (1.6), we obtain

$$\|u_t(t); L^2_x\|, \|u_{xx}(t); L^2_x\| \text{ and } \|\theta(t); L^1_x\| \leq C \qquad \text{for } 0 \leq t \leq T.$$

Then the solution obtained by Theorems 1.1 and 4.1 can be extended globally in time.

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